MATH 4010 | Tutorial 8 | 12/03/2015

- Inner Product Spaces

X : vector space over $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$)

A mapping $\langle \cdot, \cdot \rangle$ defined on $X \times X$ by $(x,y) \longmapsto \langle x,y\rangle$ satisfy

(i) $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle \quad \forall x,y,z \in X$

(ii) $\langle \alpha x, y\rangle = \alpha \langle x,y\rangle \quad \forall x,y\in X,\ \alpha\in\mathbb{K}$

(iii) $\langle x,y\rangle = \overline{\langle y,x\rangle} \quad \forall x,y\in X$

(iv) $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0$ iff $x=0$, $\forall x\in X$.

Remarks:

1°. Inner product spaces can indu(c)e norm spaces.

$\|x\| := \sqrt{\langle x,x\rangle}$ gives a norm on inner product space.

2°. Polarization identity

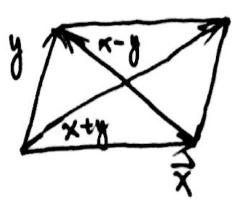
$$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) \quad \text{if } \mathbb{K}=\mathbb{R}$$

$$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) + \frac{i}{4}\left(\|x+iy\|^2 - \|x-iy\|^2\right) \quad \text{if } \mathbb{K}=\mathbb{C}$$

Qn : When does a normed space be induced by a innear product space?

Eg 1. A normed space X is induced by a inner space iff it satisfies the following parallelogram equality:

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \qquad (*)$$



Pf: $\Rightarrow$ Let X be a inner product space.

Then it is easy to check $\|x\| := \sqrt{\langle x,x\rangle}$ is a norm on X and

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle \\ &= \overline{\langle x+y, x\rangle} + \overline{\langle x+y, y\rangle} \\ &= \overline{\langle x,x\rangle + \langle y,x\rangle} + \overline{\langle x,y\rangle + \langle y, \cancel{x} y\rangle} \\ &= \overline{\langle x,x\rangle} + \overline{\langle y,x\rangle} + \overline{\langle x,y\rangle} + \langle y,y\rangle \\ &= \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2\end{aligned}$$

Similarly, $\|x-y\|^2 = \|x\|^2 - \langle x,y\rangle - \langle y,x\rangle + \|y\|^2$

So. $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

$\Leftarrow$ Let $(X, \|\cdot\|)$ be a norm space satisfying $(*)$.

Define $\langle x, y\rangle := \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) + \frac{i}{4}(\|x+iy\|^2 - \|x-iy\|^2)$

We claim that $\langle \cdot,\cdot\rangle$ is a inner product on X.

(i) $\forall\, x, y, z \in X$

$$\langle x, z\rangle + \langle y, z\rangle = \frac{1}{4}(\|x+z\|^2 - \|x-z\|^2) + \frac{i}{4}(\|x+iz\|^2 - \|x-iz\|^2)$$
$$+ \frac{1}{4}(\|y+z\|^2 - \|\cancel{x}y-z\|^2) + \frac{i}{4}(\|\cancel{x}y+iz\|^2 - \|\cancel{x}y-iz\|^2)$$

(Note that $x = \frac{x+y}{2} + \frac{x-y}{2}$ and $y = \frac{x+y}{2} - \frac{x-y}{2}$)

$$= \frac{1}{2}\left(\left\|\frac{x+y}{2}+z\right\|^2 + \left\|\frac{x-y}{2}\right\|^2\right) - \frac{1}{2}\left(\left\|\frac{x+y}{2}-z\right\|^2 + \left\|\frac{x-y}{2}\right\|^2\right)$$
$$+ \frac{i}{2}\left(\left\|\frac{x+y}{2}+iz\right\|^2 + \left\|\frac{x-y}{2}\right\|^2\right) - \frac{i}{2}\left(\left\|\frac{x+y}{2}-iz\right\|^2 + \left\|\frac{x+y}{2}\right\|^2\right)$$
$$= \frac{1}{2}\left(\left\|\frac{x+y}{2}+z\right\|^2 - \left\|\frac{x+y}{2}-z\right\|^2\right)$$
$$+ \frac{i}{2}\left(\left\|\frac{x+y}{2}+iz\right\|^2 - \left\|\frac{x+y}{2}-iz\right\|^2\right)$$
$$= \frac{1}{2}\left\langle \frac{x+y}{2}, z\right\rangle$$

Since $\langle 0, y\rangle = \frac{1}{4}(\|0+y\|^2 + \|0-y\|^2) + \frac{i}{4}(\|0+iy\|^2 - \|0-iy\|^2) = 0$

One has $\langle x, z\rangle = \frac{1}{2}\langle \frac{x}{2}, z\rangle$ which implies that

$$\langle x+y, z\rangle = \frac{1}{2}\left\langle \frac{x+y}{2}, z\right\rangle$$

So, $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

(ii) Given any $x, y \in X$, define $f(\lambda) = \langle \lambda x, y\rangle$

Then $f(\lambda_1 + \lambda_2) = \langle \lambda_1 x + \lambda_2 x, y\rangle = \langle \lambda_1 x, y\rangle + \langle \lambda_2 x, y\rangle$
$= f(\lambda_1) + f(\lambda_2)$

and f is continous since $\langle \cdot,\cdot\rangle$ defined above is continous due to the cts of norm.

Claim: $f(\lambda) = \lambda f(1)$, $\forall \lambda \in \mathbb{R}$, i.e. $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$.

Proof of claim: Since $f(\lambda_1 + \lambda_2) = f(\lambda_1) + f(\lambda_2)$, then $\forall n \in \mathbb{N}$.

$f(n\lambda) = n f(\lambda)$. Taking $\lambda = \frac{1}{n}$, $f(1) = n f(\frac{1}{n})$, i.e. $f(\frac{1}{n}) = \frac{1}{n} f(1)$

$f(\frac{m}{n}) = m f(\frac{1}{n}) = \frac{m}{n} f(1)$ $\forall n, m \in \mathbb{N}$. $f(0) = f(2\cdot 0) = 2f(0) \Rightarrow f(0) = 0$.

So, $f(q) = q f(1)$, $\forall q \in \mathbb{Q}$ be a rational number.

Note that $\mathbb{Q}$ is dense in $\mathbb{R}$ and f is cts.

So $f(\alpha) = \alpha f(1)$, $\forall \alpha \in \mathbb{R}$.

On the other hand

$$\begin{aligned}
\langle ix, y\rangle &= \frac{1}{4}(\|ix+y\|^2 - \|ix-y\|^2) + \frac{i}{4}(\|ix+iy\|^2 - \|ix-iy\|^2) \\
&= \frac{1}{4}(\|x-iy\|^2 - \|x+iy\|^2) + \frac{i}{4}(\|x+y\|^2 - \|x-y\|^2) \\
&= i\langle x, y\rangle
\end{aligned}$$

Therefore, $\forall\, c = \alpha + i\beta \in \mathbb{C}$,

$$\begin{aligned}
\langle cx, y\rangle &= \langle(\alpha+i\beta)x, y\rangle = \langle \alpha x, y\rangle + \langle i\beta x, y\rangle \\
&= \alpha\langle x, y\rangle + i\beta\langle x, y\rangle = c\langle x, y\rangle.
\end{aligned}$$

(iii)
$$\begin{aligned}
\langle y, x\rangle &= \frac{1}{4}(\|y+x\|^2 - \|y-x\|^2) + \frac{i}{4}(\|y+ix\|^2 - \|y-ix\|^2) \\
&= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) + \frac{i}{4}(\|-iy+x\|^2 - \|iy+x\|^2) \\
\Rightarrow \overline{\langle y, x\rangle} &= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) + \frac{i}{4}(\|x+iy\|^2 - \|x-iy\|^2) \\
&= \langle x, y\rangle
\end{aligned}$$

(iv)
$$\begin{aligned}
\langle x, x\rangle &= \frac{1}{4}(\|x+x\|^2 - \|x-x\|^2) + \frac{i}{4}(\|x+ix\|^2 - \|x-ix\|^2) \\
&= \frac{1}{4}\|2x\|^2 + \frac{i}{4}(\|x+ix\|^2 - \|x+ix\|^2) = \|x\|^2 \geq 0
\end{aligned}$$

and $\langle x, x\rangle = 0$ iff $\|x\| = 0$ iff $x = 0$

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Eg 2. $C[a,b]$ (with norm $\|x\| = \sup_{t\in[a,b]} \|x(t)\|$) is not a inner product space.

Pf: Choose $x(t) = 1$, $y(t) = \frac{t-a}{b-a}$ $\forall t \in [a,b]$. Then

$\|x\| = \|y\| = 1$ and $x(t) + y(t) = 1 + \frac{t-a}{b-a} \Rightarrow \|x+y\| = 2$

$x(t) - y(t) = 1 - \frac{t-a}{b-a} \Rightarrow \|x-y\| = 1$

So $\|x+y\|^2 + \|x-y\|^2 = 5$, but $2(\|x\|^2 + \|y\|^2) = 4$

The parallelogram equality is invalid, therefore $C[a,b]$ is not a inner